THE MELTING AND SOLIDIFICATION OF ICE

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INTRODUCTION

The physical process of the melting and solidification of ice can best be explained with an example; assume we have a slab of ice which is being heated from one side. there is a region between the ice and the water called the Interface which is the boundary between the two phases: the liquid phase, and the solid phase. When the ice block melts, the liquid region increases and the solid region decreases causing the position of the interface to move over time. In this project, I will be investigating further how one can mathematically model the phase change of water which can be described as a Stefan problem.

APPROACH

A Stefan problem is a moving boundary problem due to the problem's characteristic boundary, known as the interface, which is constantly moving as time goes on. J. Stefan introduced these problems in order to investigate the diffusion of heat while a substance is undergoing a phase change, e.g. water changing from its solid phase to its liquid phase. The solutions to the problems allows us to determine the location of the interface, $X(t)$, and the temperature of the solid and/or liquid. To model correctly as a Stefan problem, we must assume

- That the interface between the solid medium and the liquid medium is a surface of zero thickness in two dimensions.
- That the only heat transfer is through means of conduction.
- That the melting temperature Tm is fixed at 0℃.
- That the Latent Heat L is constant at 334 kJ/kg

THE ONE-PHASE PROBLEM

Derivation:

- Initially at t=0, the ice block is pure solid meaning that the position of the interface will be at x=0 or in other words, $X(0)=0$.
- As the interface is the region between phases, it is always at the melt temperature regardless of time and space, therefore $T(X(t),t)=T_m$
- The temperature of the liquid at the left hand boundary at any point in time will always be equal to the boundary temperature. $T(0,t)=T_L$

By solving the system of equations in a dimensionless manner, i.e. creating new variables that remove the measurement units, we can create a new dimensionless system as shown in Figure 2.

Heat conduction equation in the melt region

$$\frac{\partial T}{\partial t} = \alpha_L \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < X(t), \quad t > 0$$

Interface temperature

$$T(X(t),t) = T_m \qquad t > 0$$

Stefan condition

$$\rho_L L \frac{dX(t)}{dt} = -k_L \frac{\partial T(X(t),t)}{\partial x} \qquad t > 0$$

Initial conditions

$$X(0) = 0$$

Boundary conditions

$$T(0,t) = T_L \qquad t > 0$$

Figure 1. One-Phase system

Heat conduction equation in the melt region

$$\frac{\partial T'}{\partial t'} = \frac{\partial^2 T'}{\partial x'^2} \qquad 0 < x' < X'(t'), \quad t' > 0$$

Interface temperature

$$T'(X'(t'),t') = 0 \qquad t' > 0$$

Stefan condition

$$\frac{dX'(t')}{dt'} = -St_L \frac{\partial T'(X'(t'),t')}{\partial x'} \qquad t' > 0$$

Initial conditions

$$X'(0) = 0$$

Boundary conditions

$$T'(0,t') = 1 \qquad t' > 0$$

Figure 2. Dimensionless One-Phase

TECHNIQUE

The Stefan Problem can be split into two cases: the One-Phase case and the Two-Phase case. Unlike the two-phase case, in the one-phase case we assume that the temperature of the solid is constant and equal to the melting point, $T_m=T_s=0$.

We must also take into account the thermal properties which allow us to model the equation in a more realistic manner. c= Specific Heat Capacity, k= Thermal conductivity, ρ=Density and α= Thermal diffusivity. The higher the thermal diffusivity, the more heat diffuses outwards over time.

With the assumptions and thermal properties listed above, we can use known equations such as Heat Flux, Neumann's condition and the classic Stefan Equation to form a system of equations that will allow us to model the stages of ice melting.

RESULTS

To obtain numerical solutions to the dimensionless one-phase case, we must solve for λ using the Newton-Raphson method. The solutions are as follows:

$$\lambda_{n+1} = \lambda - \frac{g(\lambda)}{g'(\lambda)}$$

$$T'(x',t') = 1 - \frac{\operatorname{erf}\left(\frac{x'}{2\sqrt{t'}}\right)}{\operatorname{erf}(\lambda)}$$

$$X'(t') = 2\lambda\sqrt{t'} \qquad 0 < x' < X'(t')$$

$$\lambda e^{\lambda^2} \operatorname{erf}(\lambda) = -\frac{St_L}{\sqrt{\pi}} \qquad t' > 0$$

Figure 3 shows us that the interface moves further along the ice block as the time in the system is increased. This is due to the fact that the ice block is being heated from the left hand side meaning the heat is conductively transferred through to the right hand side.

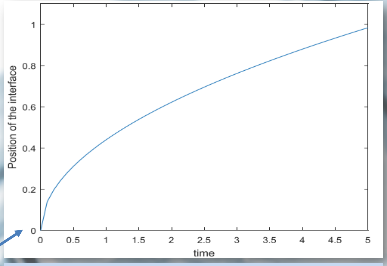

Figure 3. Interface against Time

Figure 4 shows the temperature of the liquid, T'(x',t'), increases as we travel from the right hand boundary to the left. The graph also satisfies the boundary condition which states T'(0,t')=1.

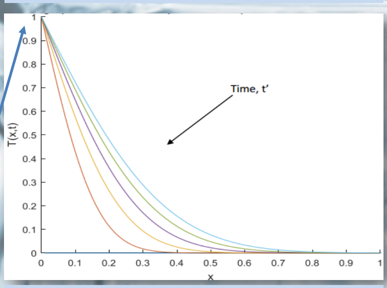

Figure 4. Temperature against x

METHOD ANALYSIS

In order to solve the dimensionless one-phase problem, we assume that there is absolutely no change in temperature in the solid region. the approach as a whole is slightly unrealistic as we assume that there is no temperature change across the solid section of the ice block when in reality, the solid may even begin to melt from another boundary. However, the results prove that the one phase problem holds and is profitable for modelling this Stefan problem.

Whilst plotting the solutions to the two-phase case, the graphical representations did not make sense as the solid side rose to a temperature of 80 which is impossible as the dimensionless solution has a positive boundary of temperature=1. I can only assume that the problem lies either with the value for lambda or one of the thermal conditions as the solution itself holds. Below are the two-phase dimensionless solutions:

$$X'(t') = 2\lambda\sqrt{t'} \qquad T_L'(x',t') = 1 - \frac{\operatorname{erf}\left(\frac{x'}{2\sqrt{t'}}\right)}{\operatorname{erf}(\lambda')}$$

$$T_S'(x',t') = -1 + \frac{1 - \operatorname{erf}\left(\frac{x'}{2\upsilon\sqrt{t'}}\right)}{1 - \operatorname{erf}\left(\frac{\lambda'}{\upsilon}\right)} \qquad \text{where } \lambda = \frac{X'(t')}{2\sqrt{t'}}$$

$$\lambda\sqrt{\pi} = \frac{St_L}{\exp(\lambda^2)\operatorname{erf}(\lambda)} - \frac{\upsilon\, St_S}{\exp\left(\frac{\lambda^2}{\upsilon^2}\right)\operatorname{erfc}\left(\frac{\lambda}{\upsilon}\right)} \qquad 0 < x' < X'(t') \qquad t' > 0$$

The two-phase case is more realistic as it takes into account the thermal conditions and the temperature changes in the solid as well as the liquid.

CONCLUSION

Overall, I admire the Stefan approach and deem it as a valid model for measuring the melting and solidification of ice.

Improvements:

- Because of the variation between thermal properties, I believe that a series of scientific experiments should be carried out in order to effectively determine the accurate values of the thermal conditions.
- A suggestion for a more complicated approach could be working on the Stefan problem to create a problem that takes into account the thermal transfer in the form of radiation, as this would make the solutions more realistic.