Applications of General Relativity to Cosmology

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Abstract
The paper discusses applications of general relativity to cosmology in the field known as 'relativistic cosmology' and the mathematical techniques and equations required (including the Einstein field equations). In the process several mathematical objects are revised and explained in a novel way including metrics and tensors and we describe the basic features of special and general relativity. In particular we focus on the Weyl postulate and geometric features of the Robertson-Walker metric. We then derive an energy-momentum tensor to describe the large-scale structure of the Universe and combine these with the afore-mentioned metric to derive the Friedmann equations. These ingredients provide us with several relativistic cosmological models which take us to the cutting edge of current observational cosmology including cosmological parameters and the size and fate of the Universe. We finish by choosing a relevant area of astrophysical research known as mass accretion whereby a gravitating body increases its mass. We describe the basic ideas of accreting systems and radiation from accretion, selecting compact binaries as an example and also accretion-powered binaries in general such as black hole binaries. Throughout we clarify the link with the mathematical techniques which we have been discussing.

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Introduction

In this paper we will discuss the applications of general relativity to astrophysics and cosmology. As the title suggests we will concentrate more on the cosmology as the astrophysics is perhaps more familiar to the general public (ie. it is fairly intuitive to grasp the idea that a concentration of mass would warp the spacetime 'fabric' surrounding it). The applications to cosmology are less familiar but we will not neglect the astrophysics either and in particular we will finish with a discussion of some fascinating astrophysical phenomena. In keeping with the theme of the conference we will also incorporate software applications where possible. Mathematical software is becoming essential in *numerical relativity* since there are still very few analytical solutions to the Einstein field equations. This is slightly worrying since in physics one could argue that the solutions to equations are more important than the equations themselves. For this reason it is necessary to study numerical approximate solutions to the field equations when working with spacetimes which are not 'straightforward'.

1. Main Discussion

We begin by revising the basic principles of special and general relativity. SR combines insights from classical mechanics and electrodynamics. We consider a bar magnet encircled by a conducting wire loop. We can keep the loop stationary and move the magnet or keep the magnet stationary and move the loop and the EMF will be the same (assuming that the relative motion is the same in both cases). Traditional electrodynamics draws a distinction between moving charges which are coupled with magnetic fields and stationary charges which are not but experiment tells us that it is the *relative* motion which determines what we observe. We must also consider *when* we observe an event. As well as choosing the coordinate system from which we observe the magnet-loop pair there must be a set of synchronized clocks allowing us to associate a particular time with an event that occurs at a particular position in the coordinate system.

Historically the other issue was the speed of light. Since electric and magnetic fields oscillate in the form of electromagnetic waves it might be a legitimate question to ask what the medium is in which the field exists. This proposed medium was known as the ether and it was thought that this could be a privileged reference frame for light to propagate in. If that were the case, it should be possible to detect slight differences in light speed along the horizontal and vertical axes of the laboratory frame as the Earth rotates through the ether but experiments continue to fail to find any significant difference in light speed depending on direction. This leads to the two postulates of SR:

1. The speed of light in a vacuum takes a fixed value in all inertial (ie. non-accelerating) reference frames.
2. The laws of physics are form-invariant in all inertial frames and the equations describing them can be written in the same form.

These two postulates are compatible if we transform the coordinates of events in different frames using the Lorentz transformations, whose effects are well-documented. The key thing is that these transformations blend space and time together whereas the earlier Galileo transformations did not.

Geometrically speaking, SR can only describe flat Minkowski spacetime. We can visualise this kind of spacetime with a Minkowski diagram. In Figure 1 the blue line through the origin with gradient 1 represents the world-line of a photon passing through the origin at time $t = 0$ and the red lines represent the ct and x axes of a frame moving away from the stationary frame with relative velocity V. The photon path is observed to be the same in both frames and it can be visualised as the edge of an hourglass-shaped lightcone belonging to event 1 at the origin. An event on (or in) the lightcone is a past or future event which could be causally related to the event at the origin. Events which lie inside the lightcone have time-like 4-vectors: they are causally related. Events on the surface of the double cone are light-like separated and have null 4-vectors.

These events with null vectors are also causally related but can be agreed upon by all inertial observers. The light cone at an event is the set of points which are linked to that event by light signals. For that reason any group of inertial observers will agree that an event on the null cone must be connected by a light signal whereas an inertial observer could perceive two time-like separated events at the same position but at different times (as the name suggests). Time-like or null vectors in or on the upper cone are future-pointing whilst those on the lower cone (in the part of the hourglass below the axes) are past-pointing. Events in the continuous region outside the lightcone have space-like 4-vectors: these events are not causally related.

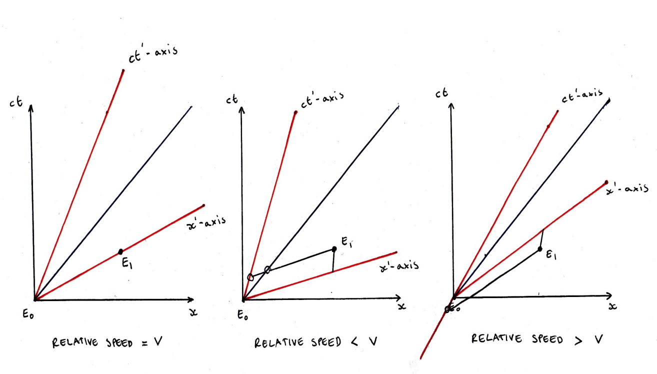

Figure 1. Minkowski diagram to demonstrate two frames in standard configuration at relative velocity V

In all three diagrams event 2 is always at the same point in the stationary frame S outside the lightcone but if we alter the relative velocity the axes for the moving frame change their gradients and it is possible for event 1 to precede event 2 in the moving frame (and vice versa). The construction lines for event 2 can intersect the ct' axis at different points so there is no causality between them. [1]

The form-invariant spacetime separation between two events in Minkowski spacetime can be written as a summation of the coordinates over dummy indices using the Minkowski metric:

$$(\Delta s)^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\, \Delta x^{\mu}\, \Delta x^{\nu}.$$

This equation shows the way that we can build a *line element* (a building block) for a geometry by relating the coefficients of the metric tensor to the coordinate intervals of the space. The Minkowski metric is simple enough.

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The metric in a sense is the key which unlocks the geometry. However, complete knowledge of the geometry of the space does not determine a unique metric since there are multiple ways of writing the metric in different coordinate systems (knowledge of the lock does not determine one unique key which fits it, if you like). Flat spacetime is a geometry of a kind but we clearly need to generalise if we wish to examine spaces which are curved and geometrically rich. The remarkable thing about this is that in generalising SR to accelerating frames we naturally move into studying curved spacetimes, since classical gravity is a consequence of spacetime curvature and the distinction between 'acceleration' and 'free-fall' is dissolved.

The basic physical principles of GR are:

1. Inertial forces cannot be distinguished from gravitational forces. The motion of bodies subject to an external gravitational field cannot be distinguished by experiment from the motion of bodies subject to uniform acceleration.
2. In order to talk about the laws of physics across curved spacetime we have to modify the equations that we use for flat spacetime so that they employ tensors and covariant rather than partial derivatives.
3. GR gives the same predictions as Newtonian gravitation under classical circumstances. The Einstein field equations describing rates of change in GR can be used to recover the Poisson equation when the gravitational field is weak and the particles are moving slowly.

The fact that the gravitational field is weak allows us to model the associated spacetime as flat spacetime, meaning that the metric tensor is close to the Minkowski metric but with an additional perturbation added to it. [2]

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \text{ where } |h_{\mu\nu}| << 1.$$

To derive the field equations of GR the metric tensor has to replace the gravitional potential of the Poisson equation because in GR gravity is a property of a differentiable spacetime manifold's geometry rather than a property of a field (or more exactly, what we perceive as gravity is the result of objects following geodesics determined by spacetime curvature). Following Principle 2 we generalise mass density to curved spacetime manifolds in the form of the energy-momentum tensor $T_{\mu\nu}$. Curved spacetime is determined by the energy and momentum of the region in the form of matter and radiation. The elements of this tensor describe the density and flow of energy and momentum in spacetime and we incorporate these into a set of PDEs known as the Einstein field equations. Solving them analytically is difficult although there are some notable solutions involving symmetry (especially spherical symmetry).

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu}.$$

The terms on the left-hand side are the 'geometric' terms. R is the Ricci scalar obtained via contraction of the Ricci tensor and it assigns a value to the distortion created by the curvature of a Riemannian manifold. Solving the Einstein equations would involve finding the metric which matches up with the stress-energy tensor whose components we hopefully know. If we find the metric, we know the geometric structure of the spacetime manifold and from there its gravitational effects. These PDEs are balanced tensor equations. It is essential in GR that equations describing physical laws are written as balanced tensor relationships in order that we know that their form is invariant under general coordinate transformations. Tensors transform between coordinate systems in a specific way. We will focus on an indicial definition of tensors as this is the one which most easily allows us to perform calculations and the more mathematical definitions involving elements of vector spaces require prior knowledge of linear algebra.

In a coordinate transformation the new coordinates are functions of the old coordinates. This is true with the Lorentz transformations but in this case the frames always move with constant relative velocity so the functions involved are linear. If the frames are not aligned in standard configuration we use partial derivatives of the coordinates, but linearity ensures that these are constants. For a general tensor (likely a 4-tensor when working in GR) the following transformation law holds:

$$T'^{\mu_1,\mu_2,\ldots,\mu_m} = \sum_{\nu_1,\nu_2,\ldots,\nu_m} \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}}, \ldots, \frac{\partial x'^{\mu_m}}{\partial x^{\nu_m}} \, T^{\nu_1,\nu_2,\ldots\nu_m}.$$

This only applies for a a contravariant tensor of arbitrary rank (indicated by a raised indice). As the axes are not necessarily aligned in GR, we also need to specify the covariant version of the tensor (indicated by a lowered indice) which transforms as follows:

$$T'_{\mu_1,\mu_2,\ldots,\mu_m} = \sum_{\nu_1,\nu_2,\ldots,\nu_m} \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}, \ldots, \frac{\partial x^{\nu_m}}{\partial x'^{\mu_m}} \, T_{\nu_1,\nu_2,\ldots\nu_m}.$$

The metric is a tensor of rank 2 with lowered indices so its components must transform as follows between coordinate systems:

$$g'_{\mu\nu} = \sum_{\alpha=0}^{3} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \sum_{\beta=0}^{3} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}.$$

The indices have been colour-coded for clarity. This may also assist with performing summations in general since it is clear which is a dummy index for the purpose of the summation and which is a free index referring to separate summations. For example, for a contravariant 4-vector under a Lorentz transformation:

$$F'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\ \nu} F^{\nu},$$

$$F'^{0} = \sum_{\nu=0}^{3} \Lambda^{0}_{\ \nu} F^{\nu} \text{ and so on.}$$

There are many pieces of computer software which are used to work with the tensor algebra of GR (several of them for use with Maple or Mathematica). These are necessary due to the complexity of metrics in curved spacetime and they can be used to perform mechanical calculations of connection coefficients, geodesic equations and components of the Riemann tensor. Although it is not as powerful as other pieces of software we use Maxima here as it is a standalone package and it also has three packages to deal with the three main ways of viewing tensors (including the algebraic interpretation which is more common in mathematical physics). Using code written by V.T. Toth we have solved the Einstein field equations under the assumption of spherical symmetry and drawn a plot of the Schwarzchild gravity well. We will not quote the entirety of the output for this code as it stretches to several pages when processed by Maxima. [3]

Having finished with these preliminaries for now we move on to cosmology. In applying GR and the Einstein field equations to cosmology we assume that GR can be applied to the entire Universe (provided that we incorporate a cosmological constant Λ or an energy-momentum tensor with a dark energy contribution described by its own tensor). We are able to make this assumption because we are discussing the Universe on a large scale (ie. on the scale of galaxy superclusters and the non-luminous voids between them). On this scale it is acceptable to model the Universe as homogeneous and isotropic. If

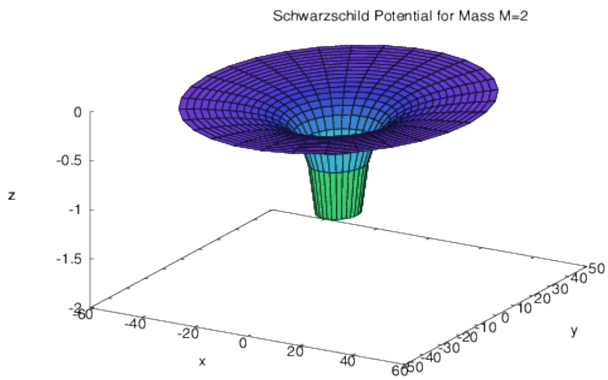

Figure 2. Plot of the Schwarzchild gravity well for a massive object

we are discussing GR globally it follows that we will need to talk about cosmic spacetime.

The Weyl postulate states that there is set of fundamental observers in cosmic spacetime who can measure proper times along their world-lines through the evolving Universe and that these proper times can be correlated to assign a cosmic time to an event (even if those observers cannot communicate). In order for this to be true the Universe must evolve in an orderly manner. The world-lines of all these fundamental observers form a smooth bundle of time-like geodesics and fundamental observers observe the CMBR to be isotropic (ie. they move with the Hubble flow). The set of all events at an instant of cosmic time forms a space-like hypersurface and the Universe as described in a cosmological model then evolves as if it were a set of successive hypersurfaces everywhere orthogonal to the world-lines. (A cosmological model is a mathematical model describing the large-scale features of the Universe). In practice we can think of a fundamental observer as an observer situated in a particular galaxy. An observer in the Milky Way is actually non-fundamental due to the galaxy's peculiar motion so we say that the CMBR is isotropic after corrections have been made to account for the motion of the galaxy. We do not consider models to which the Weyl postulate does not apply.

The path along a world-line can be specified by three spatial coordinates x^i but we must emphasise that coordinates do not need to have metrical significance in GR as they are only labels for events in a particular reference frame. In CM we can use generalized coordinates θ_1 and θ_2 to describe a double pendulum and these coordinates will have metrical significance in the sense that we can measure a coordinate interval $\mathrm{d}\theta_1$ with a protractor but this need not be the case in GR. When working with the Schwarzchild line element for the spacetime around a spherically symmetric body the difference in Schwarzchild coordinate time between two events $\mathrm{d}t$ is equal to the proper time between the events $\mathrm{d}\tau_\infty$ as measured by a stationary observer at infinity but this is fortuitous.

Bearing this in mind it is possible to define coordinates such that the world-line of the observer travelling through a family of hyperslices always has the same spatial coordinates. Coordinates of this kind are known as co-moving coordinates because they expand as the hypersurface expands. Having discarded relativistic cosmological models which are not homogeneous and isotropic we now focus on the metric which describes the spacetime assumed by the remaining models (known as the Robertson-Walker metric). The general form of the cosmic spacetime separation between events for a fundamental observer is as follows:

$$(\mathrm{d}s)^2 = c^2(\mathrm{d}t)^2 - a^2(t)\sum_{i,j=1}^{3} h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

where t is the cosmic time which would be observed by a hypothetical fundamental observer and the spatial coordinates are co-moving coordinates. The metric coefficients only depend on the spatial coordinates: the time-dependence is all contained in the scale factor $a^2(t)$. We are able to contain the cosmic time in a scaling function because homogeneity and isotropy of the model ensure that the ratios of distances between points are constant over time. Furthermore, the three-dimensional space must have constant curvature so we replace the curvature with a curvature parameter k which can take the values $+1$, 0 or -1. If we state the spatial metric in terms of co-moving polar coordinates we arrive at one of the most common forms of the RW metric:

$$(\mathrm{d}s)^2 = c^2(\mathrm{d}t)^2 - R^2(t)\left[\frac{(\mathrm{d}r)^2}{1-kr^2} + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\phi)^2\right].$$

This metric is similar to the Schwarzchild metric in that it uses spherical coordinates but in this case the coordinates are centred on the observers (ie. the observers in a particular galaxy) rather than a centre of mass. The line element is referred to interchangeably as the metric but this is not too confusing as we have seen how the line element incorporates the metric coefficients. $R(t)$ is the scale factor of the Universe so it must become larger as the Universe expands. k is linked to the global Gaussian curvature over R^2. Geometrically we might like to know how to relate the RW coordinate distance of two points to the proper distance between them (bearing in mind that proper distance changes due to expansion). We could think of the proper distance between two galaxies as the actual value which would be measured by a 'galactic ruler' at an instant of time t. By convention the co-moving distance is the proper distance measured now at time t_0. In theory the proper distance can be taken from the spatial RW metric:

$$\mathrm{d}\sigma = R(t)\sqrt{\frac{(\mathrm{d}r)^2}{1-kr^2} + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\phi)^2}.$$

We can simplify the integral involved in this calculation by considering two points on a hypersurface which are separated only by a radial co-moving coordinate such that the path between them is radial only with no contribution from θ or ϕ. This creates a standard integral which can be solved for any value of k.

$$\sigma(t) = \int_0^\gamma R(t)\frac{\mathrm{d}r}{(1-kr^2)^{\frac{1}{2}}}.$$

In particular if $k=0$ we find that the co-moving radial coordinate γ and the proper distance σ are proportional over time. The data suggest that the Universe is flat and that $k=0$ so we tend to focus on these models. A flat Universe can still be an unbounded topology whilst appearing to be bounded locally. Having $k=0$ implies that the space-like hypersurfaces are Euclidean but this does not mean that RW spacetime itself is flat since the proper distances will be changing due to the scale factor. If the scale factor is constant RW spacetime reduces to global Minkowski spacetime.

The total line-of-sight co-moving distance to a distant object is written as an integral involving the Hubble constant H_0 and the redshift z.

$$d_c = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{E(z)}.$$

$E(z)$ is a function of redshift which relates the density parameters of the Universe to the Hubble parameter via the Hubble constant.

$$E(z) = \sqrt{\Omega_{m0}(1+z)^3 + \Omega_\Lambda + \Omega_{k0}(1+z)^2}.$$

These distance definitions are mathematically sound since they are drawn from the geometric features of the RW metric but they are not necessarily distances which can be used for astronomical purposes. Observational distances generally rely on luminosity, angular diameter and proper motion and there are also some empirical relationships such as the Tully-Fisher relation for spiral galaxies. We have derived the RW metric on geometric grounds and we now seek an appropriate energy-momentum tensor. The easiest way to do this is to model the contents of cosmic spacetime as an ideal fluid. Since as fundamental observers we are observing from the instantaneous rest frame of that fluid the EM tensor can be represented by the following matrix:

$$[T^{\mu\nu}] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

These densities and pressures can be broken down into components belonging to matter, radiation and the source of dark energy. Using the RW metric and the ideal fluid EM tensor we can then write down the Einstein field equations in a form known as the Friedmann equations. These equations relate the scale factor, the curvature parameter and the densities and pressures of the fluids. If we expand to include the components as outlined above we end up with the following:

$$\left[\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right]^2 = \frac{8\pi G}{3}\left[\rho_{m0}\left[\frac{R_0}{R(t)}\right]^3 + \rho_{r0}\left[\frac{R_0}{R(t)}\right]^4 + \rho_\Lambda\right] - \frac{kc^2}{R^2},$$

$$\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}t^2} = -\frac{4\pi G}{3}\left[\rho_{m0}\left[\frac{R_0}{R(t)}\right]^3 + 2\rho_{r0}\left[\frac{R_0}{R(t)}\right]^4 - 2\rho_\Lambda\right].$$

If we assume that the scale factor has a known value R_0 at some time t_0, it is fairly easy to assume some input parameter values and so solve the first Friedmann equation to derive a scale factor for a FRW model in terms of the model's Hubble parameter at t_0. Probably the simplest of these is the de Sitter model in which $k=0$, $\rho_{m0}=0$ and $\rho_{r0}=0$. The scale factor for this model is exponential. Another historically important single-component model is the Einstein-de Sitter model in which there is only matter.

In a $k=0$ Universe the changing value of the total cosmic density always has a fixed value known as the critical density.

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}.$$

The density parameters which we mentioned earlier are expressed as fractions of the critical density. Using the Friedmann equations it can be seen that $k=0$ when $\Omega_m + \Omega_r + \Omega_\Lambda = 1$. The fact that the parameters do appear to add up to 1 is further evidence for a flat Universe. The FRW model which interests us the most is the model in which $k=0$ and $\Omega_{\Lambda 0} = \Omega_{\Lambda E}$ since this model is believed to give a good description of the Universe. In this accelerating model the rate of expansion of the Universe accelerates as a result of changing densities. A large amount of work has been done to relate the FRW models to observable quantities such as redshift. [4]

These models take us to the forefront of modern precision cosmology. This includes a determination of the time elapsed since the Big Bang and precise experimental values for cosmological parameters which tell us that the fate of the Universe is exponential expansion. In terms of the size of the observable Universe we need to exercise caution as we already know that distance is an elusive concept in RW spacetime but we can calculate the co-moving radius of the observable Universe (around $3.53c/H_0$ using the accepted values for the density parameters). The proper distances to distant objects are obviously growing and they do this at a rate measured by the Hubble parameter: this allows us to quantify the rate at which the observable Universe is increasing its size.

Much of our knowledge of cosmological parameters comes from analysing the low-amplitude anisotropies in the CMB and there are several codes which can be used to compute CMB spectra based on input parameters: of these CAMB is the most common at the time of writing. In Figure 3 we have used CAMB to calculate an image of the CMB's angular power spectrum based on accepted input parameter values. In particular the shape of the spectrum varies significantly with changes to the baryon density parameter.

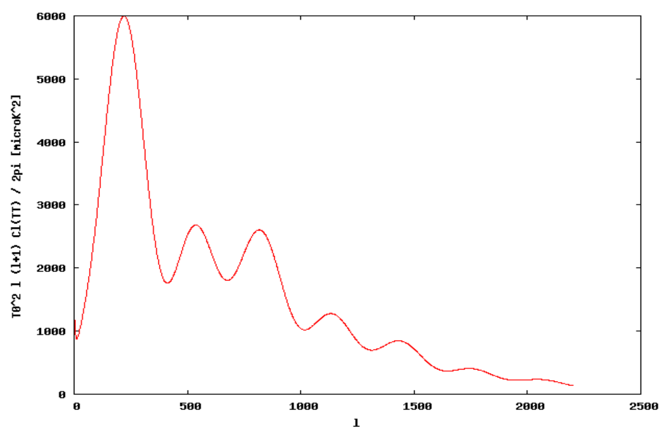

Figure 3. CAMB output showing the angular power spectrum of the CMB given typical input parameter values

The spectrum measures the variance in the CMB plotted as a function of multipole number and gives us a huge amount of information about the properties of the material which was oscillating in the early Universe. By comparing predicted spectra with the observed data it is possible to deduce the probable values of cosmological parameters along with uncertainties. The bumps in the spectrum are due to acoustic oscillations. Mathematically speaking, the power spectrum is the result of expanding the CMB structure via a spherical harmonic transform.

$$\delta(\theta,\phi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} a_{lm}Y_{lm}(\theta,\phi).$$

This is similar to a Fourier transform but spherical harmonics are orthogonal to the surface of a sphere whereas Fourier analysis requires a flat space. [5]

We finish on the topic of mass accretion. This is the process by which a gravitating body increases its mass by taking in matter from some external reservoir. Accretion releases gravitational potential energy, meaning that objects which accrete can often be powerful energy sources. As might be expected the lumonosity of an accretor is based on the equation for gravitational potential energy (bearing in mind that the accretor is accreting mass at some rate $\dot{M}$).

$$L_{\mathrm{acc}} = \frac{GM\dot{M}}{R}.$$

Accretion efficiency is highest when the accretion is onto a compact accretor (compact meaning that it is denser than a normal star). Accreting flows typically take the form of accretion discs. The surface temperature of a steady-state disc can be taken as a function of distance r from the accretor.

$$T_{\mathrm{eff}}^4(r) \approx \frac{3GM\dot{M}}{8\pi\sigma r^3}.$$

The accreting plasma heats up with decreasing distance from the accretor, emitting electromagnetic radiation in the process. The plasma is optically thick and the emission spectrum is a multi-colour black body spectrum. The disc is something like a series of rings, each one radiating locally like a black body with a temperature given by the above equation. The spectrum is the sum of all these black body spectra.

Accreting systems are rather common and there are a wide variety of binary stars in the Universe. The emitted radiation due to accretion is usually only high-energy when a compact accretor is involved and compact binaries are especially powerful emitters of high-energy radiation. The mass donor in a compact binary is generally a normal star with accretion via stellar wind or Roche-lobe overflow. There are other accretion-powered compact binaries including neutron star binaries and black hole binaries but the formation of these binaries is less well understood. [6]

2. Conclusions

We have covered the basics of an introduction to relativistic cosmology and we have also discussed an astrophysical phenomenon which requires an understanding of SR and GR when studied further. Although the RW metric must involve simplifications, GR remains as the basis of modern cosmology and the FRW models we have described take us surprisingly close to our current models for the evolution of the Universe along with density fluctuations. We have discussed some important software applications, including the essential role of computer codes in analysing the CMB and deducing cosmological parameters.

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